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USAAVLABS TECHNICAL REPORT 70-8

A FIRST APPROXIMATION THEORY TO THE EFFECTS OF TRANSVERSE SHEAR DEFORMATIONS ON THE BUCKLING AND VIBRATION OF FIBER-REINFORCED CIRCULAR CYLINDRICAL SHELLS - APPLICATION TO AXIAL COMPRESSION LOADING OF BORON- AND GLASS-EPOXY COMPOSITES

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By

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March 1970

U. S. ARMY AVIATION MATERIEL LABORATORIES
FORT EUSTIS, VIRGINIA

CONTRACT DAAJ02-68-C-0035
DEPARTMENT OF AERONAUTICS AND ASTRONAUTICS
STANFORD UNIVERSITY
STANFORD, CALIFORNIA

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This program was carried out under Contract DAAJ02-68-C-0035 with Stanford University.

The research was directed toward the development of a theory to predict the effects of transverse shear deformations on the buckling and vibration of fiber-reinforced cylindrical shells under uniform axial compression. The cases for boron-epoxy and glass-epoxy shells are presented.

The report has been reviewed by the U.S. Army Aviation Materiel Laboratories and is considered to be technically sound. It is published for the exchange of information and the stimulation of future research.

Task 1F162204A17002
Contract DAAJ02-68-C-0035
USAAVLABS Technical Report 70-8
March 1970

A FIRST APPROXIMATION THEORY TO THE EFFECTS OF TRANSVERSE SHEAR DEFORMATIONS ON THE BUCKLING AND VIBRATION OF FIBER-REINFORCED CIRCULAR CYLINDRICAL SHELLS - APPLICATION TO AXIAL COMPRESSION LOADING OF BORON- AND GLASS-EPOXY COMPOSITES

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FORT EUSTIS, VIRGINIA

SUMMARY

The effects of transverse shear deformations upon the classical buckling load of fiber-reinforced cylindrical shells under uniform axial compression have been analyzed. A stability determinant which upon evaluation yields an expression for the buckling load has been developed using a modified form of the Reissner variational principle. The buckling loads predicted by the stability determinant, with transverse shear effects neglected, have been calculated and shown to agree with previously published results for boron-epoxy and glass-epoxy cylinders. Inclusion of the transverse shear effects in the two cases investigated shows little reduction in the classical buckling loads. For general application, charts are presented to give stability criteria for fiber-reinforced composites as a function of the geometric and mechanical properties. Effective transverse shear rigidities, established by experiment, must be developed in order to estimate realistically the reduction in buckling load from the charts presented. For the determination of natural frequencies of fiber-reinforced shells, with transverse shear effects included, a frequency equation in determinant form is presented.

FOREWORD

The work reported herein constitutes a portion of a continuing effort being undertaken at Stanford University for the U.S. Army Aviation Materiel Laboratories under Contract DAAJ02-68-C-0035 (Task 1F162204A17002) to establish accurate theoretical prediction capability for the static and dynamic behavior of aircraft structural components utilizing both conventional and unconventional materials. Previous contracts supported investigations which led, in part, to the results presented in references 2, 3, 4, and 5.

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LIST OF SYMBOLS

A	shell surface area, in.^{2}
$A_{ij} = [A]$	extensional stiffness matrix for shell, lb/in.
$a_{ij} = [a]$	extensional compliance matrix for shell, in./lb
b	circumferential length of the shell, in.
$C_{ij}^{(k)} = [C^{(k)}]$	extensional stiffness matrix for k^{th} layer, lb/in.^{2}
$D_{ij} = [D]$	stiffness coupling matrix for shell, lb
$D_{ij}^{*} = [D^{*}]$	bending stiffness matrix for shell, lb-in.
$d_{ij} = [d]$	compliance coupling matrix for shell, in.
$d'_{ij} = [d']$	transpose of compliance coupling matrix for shell, in.
$d_{ij}^{*} = [d^{*}]$	modified bending stiffness matrix for shell, lb-in.
E_{11}	modulus of elasticity of fiber-reinforced composite layer in the direction of the fiber reinforcement, lb/in.^{2}
E_{22}	modulus of elasticity of fiber-reinforced composite layer in the direction perpendicular to the fiber reinforcement, lb/in.^{2}
F'	complementary energy density, lb/in.^{2}
G	shear modulus of fiber-reinforced composite layer, lb/in.^{2}
$\overline{G}_x, \overline{G}_y$	effective transverse shear rigidity in xz- and yz-planes, respectively, lb/in.^{2}
h	shell total thickness, in.

h_k	layer coordinates, in.
L	axial length of the shell, in.
L_1, L_2, L_3, L_4, L_5 $L_6, L_7, L_8, L_9, L_{10}$	linear operators defined in text
M_x, M_y, M_{xy}	moment resultants, lb-in./in.
m	number of buckle half waves in axial direction
$N = n/2$	number of buckle full waves in circumferential direction
N_x, N_y, N_{xy}	force resultants, lb/in.
N_{x_o}	applied axial load, lb/in.
$\bar{N}_{x_o}$	nondimensional applied axial load parameter
N_1, N_2, N_3	undetermined force coefficients, lb/in.
n	number of buckle half waves in the circumferential direction
Q_x, Q_y	transverse shear force resultants, lb/in.
R	radius of shell, in.
T''	kinetic energy, in.-lb
t	time, sec
t_1, t_2	limits of an arbitrary time interval, sec
U''	Reissner functional, in.-lb
u	reference surface displacement in the x-direction, in.
u_1, u_2	undetermined displacement coefficients, in.

V	volume of shell material, in.^{3}
V''	potential of the applied loading, in.-lb
v	reference surface displacement in the y-direction, in.
v_1, v_2	undetermined coefficients, in.
W	undetermined coefficient, in.
w	reference surface displacement in the z-direction, in.
x, y, z	reference surface coordinate directions
α	nondimensional parameter defined in text
β	nondimensional parameter defined in text
Γ_x, Γ_y	undetermined coefficients, in./in.
γ	nondimensional parameter defined in text
$\gamma_{xy}, \gamma_{yz}, \gamma_{xz}$	shear strains, in./in.
$\bar{\gamma}_{xz}, \bar{\gamma}_{yz}$	average transverse shear strains, in./in.
ϵ_x, ϵ_y	extensional strain components, in./in.
κ	nondimensional parameter defined in text
κ_x, κ_y	relative rotations in x- and y-directions, respectively, in.$^{-1}$
κ_{xy}	relative twist of xy-surface, in.$^{-1}$
λ_x, λ_y	nondimensional parameters defined in text
$\mu = \left(\frac{m\pi}{L}\right) / \left(\frac{n\pi}{b}\right)$	buckle aspect ratio
ν	nondimensional parameter defined in text
ν_{12}, ν_{21}	Poisson's ratios for fiber-reinforced composite layer
ξ	nondimensional parameter defined in text

ρ	mass density per unit area, lb-sec^2/in.^{3}
σ_x, σ_y	direct stress components, lb/in.^{2}
$\tau_{xy}, \tau_{yz}, \tau_{xz}$	shear stress components, lb/in.^{2}
ϕ	Airy stress function
ω	natural frequency, rad/sec
$\bar{\omega} = \rho\omega^2 R^4/d_{11}^*$	nondimensional natural frequency parameter

SUPERSCRIPTS

0	reference surface
(k)	layer
T	transpose

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INTRODUCTION

Due to their high strength-to-weight ratios, fiber-reinforced composites are increasingly being used for structural applications. In the analyses to determine the structural characteristics of fiber-reinforced beams, plates, and shells, the structural members have been assumed to be infinitely rigid with respect to transverse shear deformations. The objective of the present analysis is to investigate the effect of relaxing the rigid-transverse-shear assumption.

The usual fiber-reinforced composite structure is laminated from thin layers, each having unidirectional fiber reinforcement. For a given layer, the ratio of the inplane shear rigidity to the extensional rigidity is much lower than the corresponding ratio in a homogeneous isotropic material. Therefore, transverse shear deformations during bending in laminated fiber-reinforced structures can be expected to be much more significant than shear deformations in homogeneous isotropic structures. In the present study, an analysis which includes the effect of transverse shear deformation on the elastic stability of fiber-reinforced circular cylindrical shells subjected to uniform axial compression is developed. Applications are made to the special cases of boron-epoxy and glass-epoxy materials.

The analysis is based upon the use of the Reissner[1] variational principle. First, the indirect variational procedure is used to establish the governing equations and the general boundary conditions for the problem. Then, to calculate buckling loads, the direct variational approach (Rayleigh-Ritz method) is applied. The general buckling criteria results are given in the form of a stability determinant, and the specific problem solution results are given in the form of charts. The nature of the development is such that with very little additional effort the natural frequency equation for fiber-reinforced shells with transverse shear effects included can be obtained; the frequency equation is presented in an appendix.

GENERAL THEORY

PROBLEM STATEMENT

The general problem considered here is the investigation of the effect of transverse shear deformations on the buckling loads of circular cylindrical shells made of fiber-reinforced composite materials under uniform axial compression. The shell geometry and coordinate system used are defined in Figure 1. The fiber reinforcement within each lamina is unidirectional; however, the orientation of a given layer with respect to adjacent layers can be arbitrary. Also, the thicknesses of each of the layers may differ. The shell wall is shown in Figure 2.

REISSNER FUNCTIONAL

In the present analysis, a variational procedure is applied to the Reissner[1] form of the potential energy. As shown by Mayers et al.[2,3,4,5], the Reissner formulation and modified forms thereof are demonstrated to be better alternatives to the potential energy principle for the nonlinear analysis of plates and shells. To facilitate the extension of the present analysis to nonlinear problems and to preserve continuity, a modified Reissner principle is used for the present linear analysis. For a thin plate or shell in which normal stresses in the direction perpendicular to the surface are neglected, the Reissner functional becomes (see reference 1)

$$U'' = \int_V \left| \sigma_x \epsilon_x + \sigma_y \epsilon_y + \tau_{xy}\gamma_{xy} + \tau_{xz}\gamma_{xz} + \tau_{yz}\gamma_{yz} - F' \right| dV \tag{1}$$

where F' is the complementary energy density and is defined by

$$F' = \int_0^{\sigma_x} \epsilon_x d\sigma_x + \int_0^{\sigma_x} \epsilon_y d\sigma_y + \int_0^{\tau_{xy}} \gamma_{xy} d\tau_{xy} + \int_0^{\tau_{xz}} \gamma_{xz} d\tau_{xz} + \int_0^{\tau_{yz}} \gamma_{yz} d\tau_{yz} \tag{2}$$

In the linear elastic case,

$$F' = \frac{1}{2}\left| \sigma_x \epsilon_x + \sigma_y \epsilon_y + \tau_{xy}\gamma_{xy} + \tau_{xz}\gamma_{xz} + \tau_{yz}\gamma_{yz} \right| \tag{3}$$

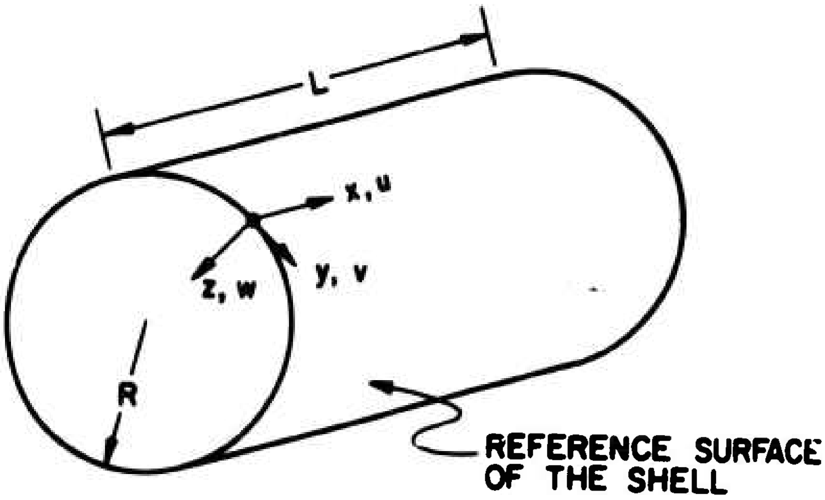

Figure 1. Sign Convention and Geometry.

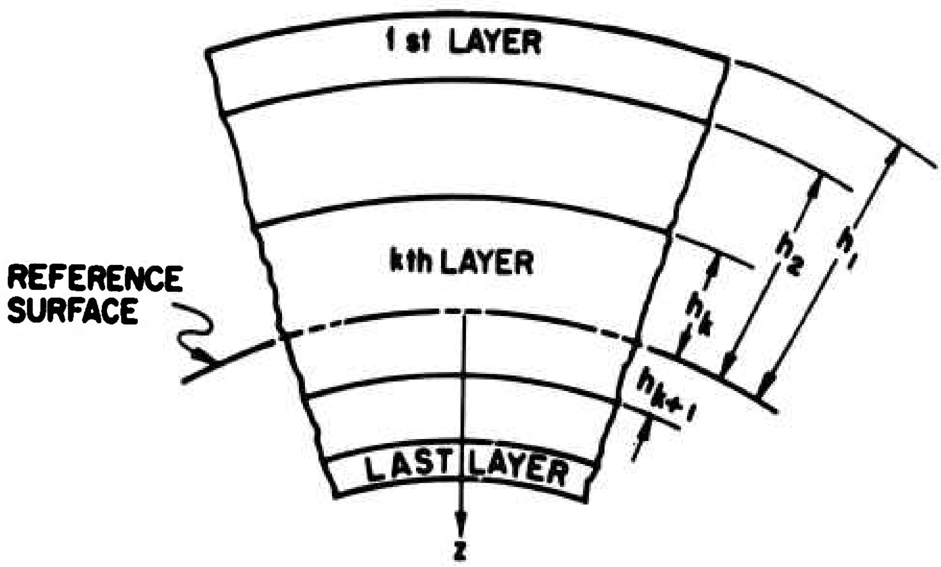

Figure 2. Element of Shell Wall.

REISSNER FUNCTIONAL – PLATES AND SHELLS

The strains can be expressed in terms of the reference-surface strains and the relative rotations (corresponding to the Kirchoff-Love assumptions) as

$$\begin{aligned} \epsilon_x &= \epsilon_x^0 - z\kappa_x \\ \epsilon_y &= \epsilon_y^0 - z\kappa_y \\ \gamma_{xy} &= \gamma_{xy}^0 - z\kappa_{xy} \end{aligned} \tag{4}$$

After integration through the thickness, the Reissner functional becomes

$$\begin{aligned} U'' = \int_A \Big| N_x\epsilon_x^0 - M_x\kappa_x + N_y\epsilon_y^0 - M_y\kappa_y + N_{xy}\gamma_{xy}^0 - M_{xy}\kappa_{xy} \\ + Q_x\bar{\gamma}_{xz} + Q_y\bar{\gamma}_{yz} - \int_h F'dz \Big| dA \end{aligned} \tag{5}$$

where the following usual definitions of the stress resultants apply:

$$\begin{aligned} N_x &= \int \sigma_x dz \\ N_y &= \int \sigma_y dz \\ N_{xy} &= \int \tau_{xy} dz \\ \\ M_x &= \int z\sigma_x dz \\ M_y &= \int z\sigma_y dz \\ M_{xy} &= \int z\tau_{xy} dz \\ \\ Q_x &= \int \tau_{xz} dz \\ Q_y &= \int \tau_{yz} dz \end{aligned} \tag{6}$$

and where $\bar{\gamma}_{xz}$ and $\bar{\gamma}_{yz}$ represent average transverse shear strains.

CONSTITUTIVE LAW

Discussions of the linear-elastic law for laminated anisotropic plates and shells can be found in references 6, 7, 8 and 9. Assumption of a state of generalized plane stress gives the stress-strain relations for the k^{th} layer as

$$\begin{Bmatrix} \sigma_x^{(k)} \\ \sigma_y^{(k)} \\ \tau_{xy}^{(k)} \end{Bmatrix} = \left[C^{(k)}\right] \begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{Bmatrix} \tag{7}$$

where the symmetrical matrix $[C^{(k)}]$ represents the elastic coefficients associated with the k^{th} lamina. The elastic coefficients depend upon the elastic properties of the lamina referred to a set of elastic axes and the orientation of the elastic axes with respect to the coordinate system of the reference surface in the structure.

The stress resultants can now be related to the reference-surface strains and the relative rotations and twist by

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix} = [A] \begin{Bmatrix} \epsilon_x^0 \\ \epsilon_y^0 \\ \gamma_{xy}^0 \end{Bmatrix} - [D] \begin{Bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix} \tag{8}$$

$$\begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} = [D] \begin{Bmatrix} \epsilon_x^0 \\ \epsilon_y^0 \\ \gamma_{xy}^0 \end{Bmatrix} - [D^*] \begin{Bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix} \tag{9}$$

The reference surface constitutive law is given now by equations (8) and (9)

where $[A]$, $[D]$, and $[D^*]$ are 3×3 matrices whose elements depend upon the physical and geometric properties of the laminae (see Appendix VI). Coupling of the reference-surface strains and the loads with the bending moments and the relative rotations and twist due to the nonsymmetrical arrangement of the laminae relative to the reference surface is produced by $[D]$. The details of the evaluation of the elements of the various matrices are given in references 6 through 9.

The transverse shear forces and average transverse shear strains are assumed to be related by

$$Q_x = h\bar{G}_x\bar{\gamma}_{xz} \quad \text{and} \quad Q_y = h\bar{G}_y\bar{\gamma}_{yz} \tag{10}$$

where $\bar{G}_x$ and $\bar{G}_y$ are the effective transverse shear rigidities.

STRAIN-DISPLACEMENT RELATIONS

For a cylindrical shell, using the Donnell[10] approximation, the reference surface strains for small deflections are given by

$$\begin{aligned} \epsilon_x^0 &= \frac{\partial u}{\partial x} \\ \epsilon_y^0 &= \frac{\partial v}{\partial y} - \frac{w}{R} \\ \epsilon_{xy}^0 &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{aligned} \tag{11}$$

In a manner analogous to that of Libove and Batdorf[11], the relative rotations and the relative twist are given by the relations

$$\begin{aligned} \kappa_x &= \frac{\partial^2 w}{\partial x^2} - \frac{\partial \bar{\gamma}_{xz}}{\partial x} \\ \kappa_y &= \frac{\partial^2 w}{\partial y^2} - \frac{\partial \bar{\gamma}_{yz}}{\partial y} \\ \kappa_{xy} &= 2\frac{\partial^2 w}{\partial x \partial y} - \frac{\partial \bar{\gamma}_{xz}}{\partial y} - \frac{\partial \bar{\gamma}_{yz}}{\partial x} \end{aligned} \tag{12}$$

in which the terms involving derivatives of $\bar{\gamma}_{xz}$ and $\bar{\gamma}_{yz}$ (the average transverse shear strains) are corrections to the usual curvature terms introduced to account for the effects of transverse shear deformations.

MODIFIED-REISSNER FUNCTIONAL

In a manner similar to that of references 2, 3, 4, and 5, the Reissner functional of equation (5) is now modified so that the force resultants, the reference surface displacements, and the average transverse shear strains are the only variationally independent quantities. The bending and twisting moment resultants and the transverse shear force resultants are eliminated from the Reissner functional through the use of appropriate constitutive relations.

To achieve the modification, equation (8) is inverted and substituted into equation (9) with the result that

$$\begin{Bmatrix} \epsilon_x^0 \\ \epsilon_y^0 \\ \gamma_{xy}^0 \end{Bmatrix} = [a] \begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix} + [d'] \begin{Bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix} \tag{13}$$

$$\begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} = [d] \begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix} - [d^*] \begin{Bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix} \tag{14}$$

where $[a]$, $[d']$, $[d]$, and $[d^*]$ are 3×3 matrices which are defined in terms of $[A]$, $[D]$, and $[D^*]$ in Appendix VI. The complementary energy per unit surface area is calculated by integrating equation (3) across the thickness; by applying the definitions of equations (6); and by eliminating the transverse shear resultants, reference-surface strains, and bending moments through the use of equations (10), (13), and (14). Such a procedure – the same as that

used by Khot[9] for evaluating the strain energy — yields

$$\int F'dz = \frac{1}{2}\Big| a_{11}N_x^2 + a_{22}N_y^2 + a_{66}N_{xy}^2 + 2a_{12}N_xN_y + 2a_{16}N_xN_{xy} + 2a_{26}N_yN_{xy}$$

$$+ d_{11}^*\kappa_x^2 + d_{22}^*\kappa_y^2 + d_{66}^*\kappa_{xy}^2 + 2d_{12}^*\kappa_x\kappa_y + 2d_{16}^*\kappa_x\kappa_{xy} + 2d_{26}^*\kappa_y\kappa_{xy}$$

$$+ \bar{G}_x h\bar{\gamma}_{xz}^2 + \bar{G}_y h\bar{\gamma}_{yz}^2 \Big| \tag{15}$$

The potential associated with the applied axial compression load is given by (see reference 12)

$$V'' = -\frac{1}{2}\int_A \Big| N_{x_o}\left(\frac{\partial w}{\partial x}\right)^2 \Big| dA \tag{16}$$

Combination of equations (5), (10), (11), (12), (14), (15), and (16) gives the following modified form of the Reissner total functional to be used in the variational procedures of the remainder of this analysis.

$$U'' + V'' = \iint_A \Bigg\{ N_x\frac{\partial u}{\partial x} + N_y\left(\frac{\partial v}{\partial y} - \frac{w}{R}\right) + N_{xy}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) + \frac{1}{2}\bar{G}_x h\bar{\gamma}_{xz}^2 + \frac{1}{2}\bar{G}_y h\bar{\gamma}_{yz}^2$$

$$- (d_{11}N_x + d_{12}N_y + d_{16}N_{xy})\left(\frac{\partial^2 w}{\partial x^2} - \frac{\partial\bar{\gamma}_{xz}}{\partial x}\right)$$

$$- (d_{21}N_x + d_{22}N_y + d_{26}N_{xy})\left(\frac{\partial^2 w}{\partial y^2} - \frac{\partial\bar{\gamma}_{yz}}{\partial y}\right)$$

$$- (d_{61}N_x + d_{62}N_y + d_{66}N_{xy})\left(2\frac{\partial^2 w}{\partial x\partial y} - \frac{\partial\bar{\gamma}_{xz}}{\partial y} - \frac{\partial\bar{\gamma}_{yz}}{\partial x}\right)$$

$$- \frac{1}{2}(a_{11}N_x^2 + a_{22}N_y^2 + a_{66}N_{xy}^2 + 2a_{12}N_xN_y + 2a_{16}N_xN_{xy} + 2a_{26}N_yN_{xy})$$

$$+ \frac{1}{2}\Bigg[d_{11}^*\left(\frac{\partial^2 w}{\partial x^2} - \frac{\partial\bar{\gamma}_{xz}}{\partial x}\right)^2 + d_{22}^*\left(\frac{\partial^2 w}{\partial y^2} - \frac{\partial\bar{\gamma}_{yz}}{\partial y}\right)^2 + d_{66}^*\left(2\frac{\partial^2 w}{\partial x\partial y} - \frac{\partial\bar{\gamma}_{xz}}{\partial y} - \frac{\partial\bar{\gamma}_{yz}}{\partial x}\right)^2$$

(Continued)

$$+ 2d_{12}^{*}\left(\frac{\partial^2 w}{\partial x^2} - \frac{\partial \bar{\gamma}_{xz}}{\partial x}\right)\left(\frac{\partial^2 w}{\partial y^2} - \frac{\partial \bar{\gamma}_{yz}}{\partial y}\right)$$

$$+ 2d_{16}^{*}\left(\frac{\partial^2 w}{\partial x^2} - \frac{\partial \bar{\gamma}_{yz}}{\partial y}\right)\left(2\frac{\partial^2 w}{\partial x \partial y} - \frac{\partial \bar{\gamma}_{xz}}{\partial y} - \frac{\partial \bar{\gamma}_{yz}}{\partial x}\right)$$

$$+ 2d_{26}^{*}\left(\frac{\partial^2 w}{\partial y^2} - \frac{\partial \bar{\gamma}_{yz}}{\partial y}\right)\left(2\frac{\partial^2 w}{\partial x \partial y} - \frac{\partial \bar{\gamma}_{xz}}{\partial y} - \frac{\partial \bar{\gamma}_{yz}}{\partial x}\right)\Bigg]\Bigg\} dxdy$$

$$+ \frac{1}{2}\iint_A \left\{ N_{x_o}\left(\frac{\partial w}{\partial x}\right)^2 \right\} dxdy \tag{17}$$

EULER EQUATIONS

An indirect variational approach to the problem results in the establishment of the governing equations (Euler equations) and the associated boundary conditions of the surface. The details of the indirect approach are given in Appendix I; the resulting Euler equations are

$$\begin{aligned} \frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} &= 0 \\ \frac{\partial N_y}{\partial y} + \frac{\partial N_{xy}}{\partial x} &= 0 \end{aligned} \tag{18}$$

$$\begin{Bmatrix} \epsilon_x^0 \\ \epsilon_y^0 \\ \gamma_{xy}^0 \end{Bmatrix} = [a] \begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix} + [d'] \begin{Bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix} \tag{19}$$

$$Q_x = \frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y}$$

$$Q_y = \frac{\partial M_y}{\partial y} + \frac{\partial M_{xy}}{\partial x} \tag{20}$$

$$\frac{\partial^2 M_x}{\partial x^2} + 2\,\frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} = -\frac{N_y}{R} + N_{x_o}\,\frac{\partial^2 w}{\partial x^2} \tag{21}$$

which apply to flat $(R \rightarrow \infty)$ and circularly curved plates as well as to circular cylinders. Equations (18) represent equilibrium in the x- and y-directions, respectively. Equation (19) represents stress-displacement compatibility. Equations (20) represent moment equilibrium about the x- and y-axes, respectively. Finally, equation (21) represents equilibrium in the direction normal to the reference surface.

METHOD OF SOLUTION

ASSUMED SOLUTION FUNCTIONS

The generalized displacements and forces satisfying the compatibility conditions derived in Appendix II and Appendix III and, at least, the geometric boundary conditions (simple support for the present problem) are assumed to be

$$
\begin{aligned}
w &= W \sin\frac{m\pi x}{L}\sin\frac{n\pi y}{b} \\
\bar{\gamma}_{xz} &= \Gamma_x \cos\frac{m\pi x}{L}\sin\frac{n\pi y}{b} \\
\bar{\gamma}_{yz} &= \Gamma_y \sin\frac{m\pi x}{L}\cos\frac{n\pi y}{b}
\end{aligned}
\tag{22}
$$

$$
\begin{aligned}
u &= u_1 \sin\frac{m\pi x}{L}\cos\frac{n\pi y}{b} + u_2 \cos\frac{m\pi x}{L}\sin\frac{n\pi y}{b} \\
v &= v_1 \sin\frac{m\pi x}{L}\cos\frac{n\pi y}{b} + v_2 \cos\frac{m\pi x}{L}\sin\frac{n\pi y}{b}
\end{aligned}
\tag{23}
$$

$$
\begin{aligned}
N_x &= N_{x_o} + N_3 \sin\frac{m\pi x}{L}\sin\frac{n\pi y}{b} \\
N_y &= N_1 \sin\frac{m\pi x}{L}\sin\frac{n\pi y}{b} \\
N_{xy} &= N_2 \cos\frac{m\pi x}{L}\cos\frac{n\pi y}{b}
\end{aligned}
\tag{24}
$$

where W , Γ_x , Γ_y , u_1 , u_2 , v_1 , v_2 , N_1 , N_2 , and N_3 become the independent quantities in the direct variational procedure.

STABILITY DETERMINANT DEVELOPMENT

Substitution of the assumed functions into equation (17), integration over the prescribed limits, and subsequent variation with respect to the 9 independent quantities yields a set of 9 linear homogeneous equations. The determinant of the coefficients of the homogeneous equations establishes the condition for equilibrium in the buckled state. An expression for the applied axial load, N_{x_o} , required for equilibrium in the buckled shape as a function of the shell geometric and elastic properties, the number of circumferential waves (N) , and the buckle aspect ratio (μ) results when the determinant is evaluated. For a given shell, the buckling load corresponds to the lowest value of N_{x_o} for all the possible combinations of N and μ. The stability determinant as well as the details of the buckling load calculation is given in Appendix IV.

RESULTS AND DISCUSSION

The primary result of the present analysis is the establishment of the stability determinant given by equation (40) in Appendix IV. When the shell geometry and composite material construction are defined, the axial buckling load for a circular cylindrical shell made from any number of arbitrarily orientated fiber-reinforced layers can be determined through numerical evaluation of the stability determinant.

Specifically, the effects of transverse shear deformation on the axial buckling load of fiber-reinforced shells made of two different matrix-fiber combinations have been calculated using equation (40). Both of the shells considered have a 6.0 in. mean radius and are laminated from 0.012 in. thick layers with all fiber reinforcement oriented in the longitudinal direction. The lamina elastic properties for the two cases studied are the same as those used by Khot[8,9] and are listed below.

Boron-Epoxy Composite

$$E_{11} = 40.0 \times 10^6 \text{ lb/in.}^2 = C_{11}(1-\nu_{12}\nu_{21})$$

$$E_{22} = 4.5 \times 10^6 \text{ lb/in.}^2 = C_{22}(1-\nu_{12}\nu_{21})$$

$$\nu_{12} = 0.25 \qquad \nu_{21} = 0.028$$

$$G = 1.5 \times 10^6 \text{ lb/in.}^2 = C_{66}$$

Glass-Epoxy Composite

$$E_{11} = 7.5 \times 10^6 \text{ lb/in.}^2 = C_{11}(1-\nu_{12}\nu_{21})$$

$$E_{22} = 3.5 \times 10^6 \text{ lb/in.}^2 = C_{22}(1-\nu_{12}\nu_{21})$$

$$\nu_{12} = 0.25 \qquad \nu_{21} = 0.18$$

$$G = 1.25 \times 10^6 \text{ lb/in.}^2 = C_{66}$$

The axial buckling loads for the cases of infinite shear rigidity (no transverse shear effect), along with the reductions in the axial buckling

loads due to finite transverse shear deformations, are shown in Figures 3 and 4 for boron-epoxy and glass-epoxy composites, respectively. The axial buckling loads given in Figures 3 and 4 for the cases of 3-layer shells are the same as those calculated by Khot[8] on the basis of infinite transverse shear rigidity of the medium.

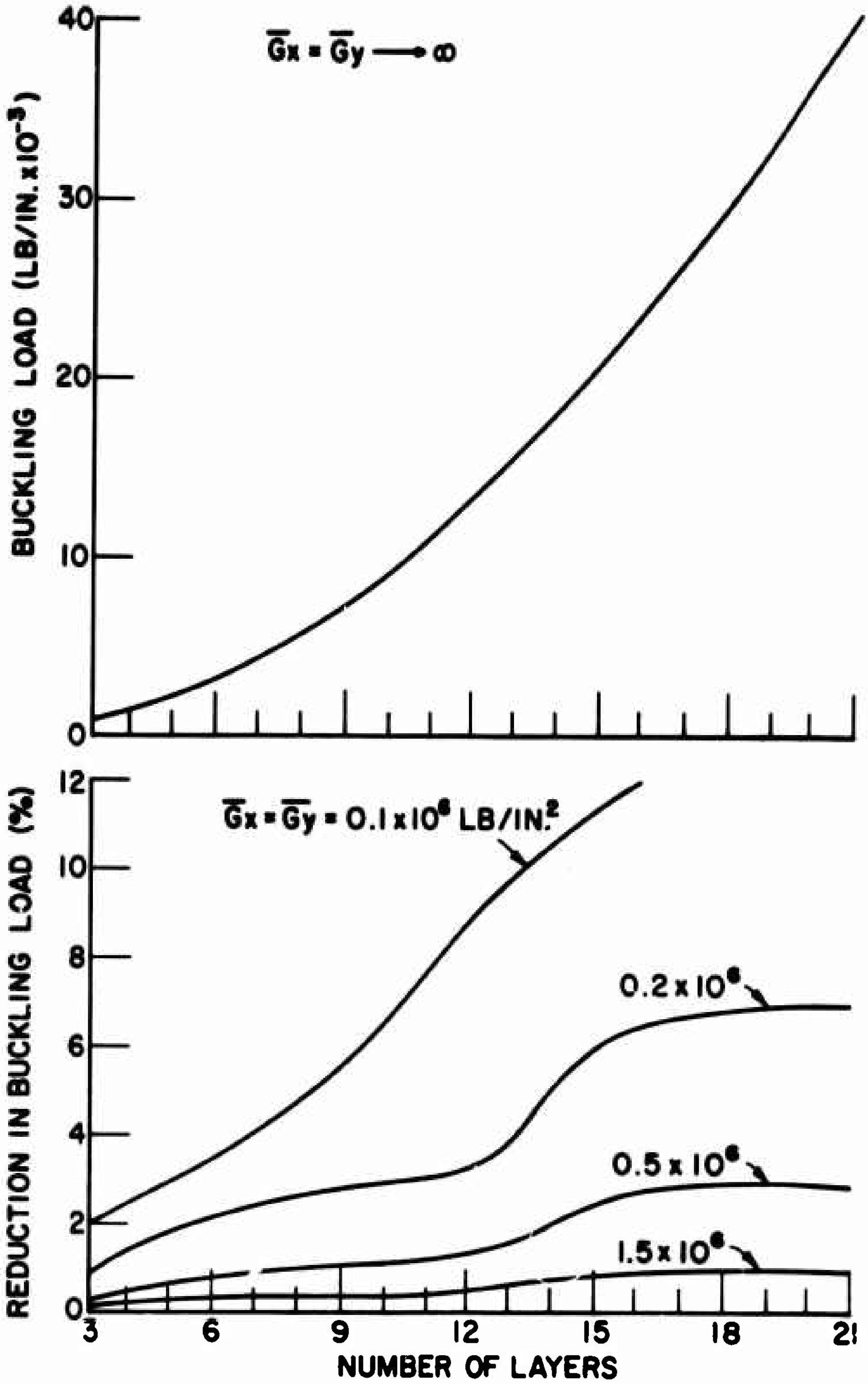

Figure 3. Stability Criteria, Including Transverse Shear Effects, for Boron-Epoxy Cylindrical Shells (R = 6.0 in., h_i = 0.012 in.)

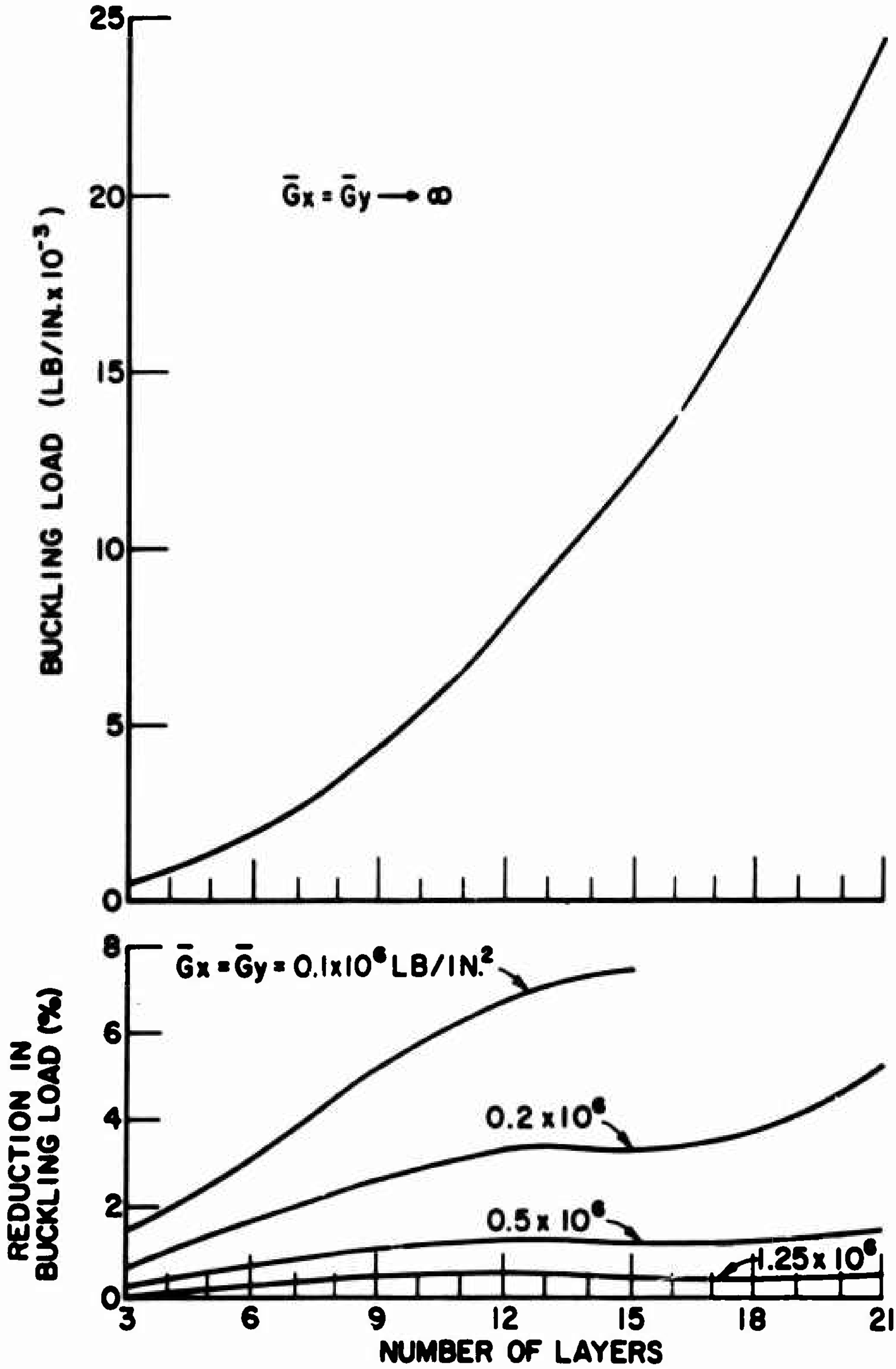

Figure 4. Stability Criteria, Including Transverse Shear Effects, for Glass-Epoxy Cylindrical Shells ($R = 6.0$ in., $h_1 = 0.012$ in.)

CONCLUDING REMARKS

For a first approximation to the effects of transverse shear deformation on the buckling load of axially compressed boron-epoxy and glass-epoxy fiber-reinforced shells, the effective shear rigidities ($\bar{G}_x$ and $\bar{G}_y$) can be assumed to have values between the shear modulus of the epoxy matrix (about 200,000 lb/in.^{2}) and the inplane shear modulus of the composite layer (1.25×10^6 lb/in.^{2} for glass-epoxy and 1.5×10^6 lb/in.^{2} for boron-epoxy). Figures 3 and 4 show that in this range of values, the buckling loads calculated with transverse shear deformations included are less than 7 percent below those calculated with the assumption of no transverse shear deformation. Actual effective transverse shear rigidities can best be determined by experiment.

Due to the similarity in the formulation of the free lateral vibration problem and the buckling problem to which most of the preceding pages have been devoted, a frequency equation which may be used to establish the effects of transverse shear deformation on the natural frequencies of circular cylindrical fiber-reinforced shells has been developed in Appendix V. For given materials, natural frequencies can be established by finding the eigenvalues of the determinant (equation 47) in a procedure similar to that used for the buckling problems.

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APPENDIX I

EULER EQUATIONS AND BOUNDARY CONDITIONS DERIVED FROM THE VARIATIONAL PRINCIPLE

The vanishing of the first variation of equation (17) with respect to N_x , N_y , N_{xy} , u , v , w , $\bar{\gamma}_{xz}$, and $\bar{\gamma}_{yz}$ gives

$$
\begin{aligned}
\delta(U'' + V'') = \iint \Bigg\{ & N_x \frac{\partial \delta u}{\partial x} + \delta N_x \frac{\partial u}{\partial x} + N_y \frac{\partial \delta v}{\partial y} - \frac{N_y}{R}\,\delta w + \delta N_y \left(\frac{\partial v}{\partial y} - \frac{w}{R} \right) \\
& + N_{xy} \frac{\partial \delta u}{\partial y} + N_{xy} \frac{\partial \delta v}{\partial x} + \delta N_{xy} \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) + \bar{G}_x h \bar{\gamma}_{xz} \delta\bar{\gamma}_{xz} + \bar{G}_y h \bar{\gamma}_{yz} \delta\bar{\gamma}_{yz} \\
& - (d_{11}N_x + d_{12}N_y + d_{16}N_{xy}) \frac{\partial^2 \delta w}{\partial x^2} + (d_{11}N_x + d_{12}N_y + d_{16}N_{xy}) \frac{\partial \delta\bar{\gamma}_{xz}}{\partial x} \\
& - d_{11}\delta N_x \left(\frac{\partial^2 w}{\partial x^2} - \frac{\partial \bar{\gamma}_{xz}}{\partial x} \right) - d_{12}\delta N_y \left(\frac{\partial^2 w}{\partial x^2} - \frac{\partial \bar{\gamma}_{xz}}{\partial x} \right) - d_{16}\delta N_{xy} \left(\frac{\partial^2 w}{\partial x^2} - \frac{\partial \bar{\gamma}_{xz}}{\partial x} \right) \\
& - (d_{21}N_x + d_{22}N_y + d_{26}N_{xy}) \frac{\partial^2 \delta w}{\partial y^2} + (d_{21}N_x + d_{22}N_y + d_{26}N_{xy}) \frac{\partial \delta\bar{\gamma}_{yz}}{\partial y} \\
& - d_{21}\delta N_x \left(\frac{\partial^2 w}{\partial y^2} - \frac{\partial \bar{\gamma}_{yz}}{\partial y} \right) - d_{22}\delta N_y \left(\frac{\partial^2 w}{\partial y^2} - \frac{\partial \bar{\gamma}_{yz}}{\partial y} \right) - d_{26}\delta N_{xy} \left(\frac{\partial^2 w}{\partial y^2} - \frac{\partial \bar{\gamma}_{yz}}{\partial y} \right) \\
& - (d_{61}N_x + d_{62}N_y + d_{66}N_{xy}) \left(2 \frac{\partial^2 \delta w}{\partial x \partial y} \right) + (d_{61}N_x + d_{62}N_y + d_{66}N_{xy}) \left(\frac{\partial \delta\bar{\gamma}_{xz}}{\partial y} \right) \\
& + (d_{61}N_x + d_{62}N_y + d_{66}N_{xy}) \left(\frac{\partial \delta\bar{\gamma}_{yz}}{\partial x} \right) \\
& - \left(2 \frac{\partial^2 w}{\partial x \partial y} - \frac{\partial \bar{\gamma}_{xz}}{\partial y} - \frac{\partial \bar{\gamma}_{yz}}{\partial x} \right) (d_{61}\delta N_x + d_{62}\delta N_y + d_{66}\delta N_{xy})
\end{aligned}
$$

(Continued)

$$- a_{11}N_x\delta N_x - a_{22}N_y\delta N_y - a_{66}N_{xy}\delta N_{xy} - a_{12}N_x\delta N_y - a_{12}\delta N_x N_y$$

$$- a_{16}N_x\delta N_{xy} - a_{16}\delta N_x N_{xy} - a_{26}N_y\delta N_{xy} - a_{26}\delta N_y N_{xy}$$

$$+ d_{11}^{*}\left(\frac{\partial^2 w}{\partial x^2} - \frac{\partial\bar{\gamma}_{xz}}{\partial x}\right)\frac{\partial^2\delta w}{\partial x^2} - d_{11}^{*}\left(\frac{\partial^2 w}{\partial x^2} - \frac{\partial\bar{\gamma}_{xz}}{\partial x}\right)\frac{\partial\delta\bar{\gamma}_{xz}}{\partial x}$$

$$+ d_{22}^{*}\left(\frac{\partial^2 w}{\partial y^2} - \frac{\partial\bar{\gamma}_{yz}}{\partial y}\right)^2\frac{\partial^2\delta w}{\partial y^2} - d_{22}^{*}\left(\frac{\partial^2 w}{\partial y^2} - \frac{\partial\bar{\gamma}_{yz}}{\partial y}\right)\frac{\partial\delta\bar{\gamma}_{yz}}{\partial y}$$

$$+ 2d_{66}^{*}\left(2\frac{\partial^2 w}{\partial x\partial y} - \frac{\partial\bar{\gamma}_{xz}}{\partial y} - \frac{\partial\bar{\gamma}_{yz}}{\partial x}\right)\frac{\partial^2\delta w}{\partial x\partial y} - d_{66}^{*}\left(2\frac{\partial^2 w}{\partial x\partial y} - \frac{\partial\bar{\gamma}_{xz}}{\partial y} - \frac{\partial\bar{\gamma}_{yz}}{\partial x}\right)\frac{\partial\delta\bar{\gamma}_{xz}}{\partial y}$$

$$- d_{66}^{*}\left(2\frac{\partial^2 w}{\partial x\partial y} - \frac{\partial\bar{\gamma}_{xz}}{\partial y} - \frac{\partial\bar{\gamma}_{yz}}{\partial x}\right)\frac{\partial\delta\bar{\gamma}_{yz}}{\partial x}$$

$$+ d_{12}^{*}\left(\frac{\partial^2 w}{\partial x^2} - \frac{\partial\bar{\gamma}_{xz}}{\partial x}\right)\left(\frac{\partial^2\delta w}{\partial y^2} - \frac{\partial\delta\bar{\gamma}_{yz}}{\partial y}\right) + d_{12}^{*}\left(\frac{\partial^2\delta w}{\partial x^2} - \frac{\partial\delta\bar{\gamma}_{xz}}{\partial y^2}\right)\left(\frac{\partial^2 w}{\partial y^2} - \frac{\partial\bar{\gamma}_{yz}}{\partial y}\right)$$

$$+ d_{16}^{*}\left(\frac{\partial^2 w}{\partial x^2} - \frac{\partial\bar{\gamma}_{xz}}{\partial x}\right)\left(2\frac{\partial^2\delta w}{\partial x\partial y} - \frac{\partial\delta\bar{\gamma}_{xz}}{\partial y} - \frac{\partial\delta\bar{\gamma}_{yz}}{\partial x}\right)$$

$$+ d_{16}^{*}\left(\frac{\partial^2\delta w}{\partial x^2} - \frac{\partial\delta\bar{\gamma}_{xz}}{\partial x}\right)\left(2\frac{\partial^2 w}{\partial x\partial y} - \frac{\partial\bar{\gamma}_{xz}}{\partial y} - \frac{\partial\bar{\gamma}_{yz}}{\partial x}\right)$$

$$+ d_{26}^{*}\left(\frac{\partial^2 w}{\partial y^2} - \frac{\partial\bar{\gamma}_{yz}}{\partial y}\right)\left(2\frac{\partial^2\delta w}{\partial x\partial y} - \frac{\partial\delta\bar{\gamma}_{xz}}{\partial y} - \frac{\partial\delta\bar{\gamma}_{yz}}{\partial x}\right)$$

$$+ d_{26}^{*}\left(\frac{\partial^2\delta w}{\partial y^2} - \frac{\partial\delta\bar{\gamma}_{yz}}{\partial y}\right)\left(2\frac{\partial^2 w}{\partial x\partial y} - \frac{\partial\bar{\gamma}_{xz}}{\partial y} - \frac{\partial\bar{\gamma}_{yz}}{\partial x}\right)$$

(Continued)

$$- N_{x_o} \frac{\partial w}{\partial x} \frac{\partial \delta w}{\partial x} \Bigg\} dxdy$$

$$= \quad 0 \qquad (25)$$

After integration by parts, regrouping of terms, and application of equations (11) and (12) along with the constitutive relations (10) and (14), equation (25) becomes

$$\delta(U'' + V'') = -\iint \left\{ \frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} \right\} \delta u dxdy - \iint \left\{ \frac{\partial N_y}{\partial y} + \frac{\partial N_{xy}}{\partial x} \right\} \delta v dxdy$$

$$+ \iint \left| \epsilon_x - a_{11}N_x - a_{12}N_y - a_{16}N_{xy} - d_{11}\kappa_x - d_{21}\kappa_y - d_{61}\kappa_{xy} \right| \delta N_x dxdy$$

$$+ \iint \left| \epsilon_y - a_{12}N_x - a_{22}N_y - a_{26}N_{xy} - d_{12}\kappa_x - d_{22}\kappa_y - d_{62}\kappa_{xy} \right| \delta N_y dxdy$$

$$+ \iint \left| \epsilon_{xy} - a_{16}N_x - a_{26}N_y - a_{66}N_{xy} - d_{16}\kappa_x - d_{26}\kappa_y - d_{66}\kappa_{xy} \right| \delta N_{xy} dxdy$$

$$+ \iint \left\{ Q_x - \frac{\partial M_x}{\partial x} - \frac{\partial M_{xy}}{\partial y} \right\} \delta \bar{\gamma}_{xz} dxdy + \iint \left\{ Q_y - \frac{\partial M_y}{\partial y} - \frac{\partial M_{xy}}{\partial x} \right\} \delta \bar{\gamma}_{yz} dxdy$$

$$+ \iint \left\{ - \frac{\partial^2 M_x}{\partial x^2} - 2 \frac{\partial^2 M_{xy}}{\partial x \partial y} - \frac{\partial^2 M_y}{\partial y^2} - \frac{N_y}{R} + N_{x_o} \frac{\partial^2 w}{\partial x^2} \right\} \delta w dxdy$$

$$+ \int_0^b \left\{ N_x \delta u \Big|_0^L \right\} dy + \int_0^L \left\{ N_{xy} \delta u \Big|_0^b \right\} dx + \int_0^L \left\{ N_y \delta v \Big|_0^b \right\} dx + \int_0^b \left\{ N_{xy} \delta v \Big|_0^L \right\} dy$$

(Continued)

$$+\int_0^b \left\{ M_x \delta \left(\bar{\gamma}_{xz} - \frac{\partial w}{\partial x} \right) \Bigg|_0^L \right\} dy + \int_0^L \left\{ M_{xy} \delta \left(\bar{\gamma}_{xz} - \frac{\partial w}{\partial x} \right) \Bigg|_0^b \right\} dx$$

$$+\int_0^L \left\{ M_y \delta \left(\bar{\gamma}_{yz} - \frac{\partial w}{\partial y} \right) \Bigg|_0^b \right\} dx + \int_0^b \left\{ M_{xy} \delta \left(\bar{\gamma}_{yz} - \frac{\partial w}{\partial y} \right) \Bigg|_0^L \right\} dy$$

$$+\int_0^b \left\{ \left[-N_{x_o} \frac{\partial w}{\partial x} + \frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} \right] \delta w \Bigg|_0^L \right\} dy$$

$$+\int_0^L \left\{ \left[\frac{\partial M_y}{\partial y} + \frac{\partial M_{xy}}{\partial x} \right] \delta w \Bigg|_0^b \right\} dx = 0 \qquad (26)$$

For the above expression to vanish for the specified arbitrary states of stress and displacement consistent with the geometric boundary conditions, each of the terms must vanish independently. The first eight terms of equation (26) lead to the Euler equations and the remaining terms establish the boundary conditions, as follows:

Inplane equilibrium:

$$\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = 0$$

$$\frac{\partial N_y}{\partial y} + \frac{\partial N_{xy}}{\partial x} = 0 \qquad (27)$$

Constitutive relation:

$$\begin{Bmatrix} \epsilon_x^0 \\ \epsilon_y^0 \\ \gamma_{xy}^0 \end{Bmatrix} = [a] \begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix} + [d'] \begin{Bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix} \qquad (28)$$

Moment equilibrium:

$$Q_x = \frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y}$$

$$Q_y = \frac{\partial M_y}{\partial y} + \frac{\partial M_{xy}}{\partial x} \tag{29}$$

Lateral equilibrium:

$$\frac{\partial^2 M_x}{\partial x^2} + 2\frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} = -\frac{N_y}{R} + N_{x_o}\frac{\partial^2 w}{\partial x^2} \tag{30}$$

Boundary conditions:

$$\begin{array}{llllll}
N_x = 0 & \text{or} & u = 0 & \text{at} & x = 0 \;, & x = L \\
N_{xy} = 0 & \text{or} & u = 0 & \text{at} & y = 0 \;, & y = b \\
N_y = 0 & \text{or} & v = 0 & \text{at} & y = 0 \;, & y = b \\
N_{xy} = 0 & \text{or} & v = 0 & \text{at} & x = 0 \;, & x = L \\
M_x = 0 & \text{or} & \bar{\gamma}_{xz} - \frac{\partial w}{\partial x} = 0 & \text{at} & x = 0 \;, & x = L \\
M_{xy} = 0 & \text{or} & \bar{\gamma}_{xz} - \frac{\partial w}{\partial x} = 0 & \text{at} & y = 0 \;, & y = b \\
M_y = 0 & \text{or} & \bar{\gamma}_{yz} - \frac{\partial w}{\partial y} = 0 & \text{at} & y = 0 \;, & y = b \\
M_{xy} = 0 & \text{or} & \bar{\gamma}_{yz} - \frac{\partial w}{\partial y} = 0 & \text{at} & x = 0 \;, & x = L \\
-N_{x_o}\frac{\partial w}{\partial x} + Q_x = 0 & \text{or} & w = 0 & \text{at} & x = 0 \;, & x = L \\
Q_y = 0 & \text{or} & w = 0 & \text{at} & y = 0 \;, & y = b
\end{array} \tag{31}$$

APPENDIX II

COMPATIBILITY OF THE RESULTANT LOADS N_x, N_y, AND N_{xy}

From Appendix I, : equilibrium conditions for the x- and y-directions are given by equations (27) as

$$\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = 0$$

$$\frac{\partial N_y}{\partial y} + \frac{\partial N_{xy}}{\partial y} = 0$$

A constitutive relation is given by equation (8) as

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix} = [A] \begin{Bmatrix} \epsilon_x^0 \\ \epsilon_y^0 \\ \gamma_{xy}^0 \end{Bmatrix} - [D] \begin{Bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix}$$

and equations (11) and (12) give

$$\epsilon_x^0 = \frac{\partial u}{\partial x}$$

$$\epsilon_y^0 = \frac{\partial v}{\partial y} - \frac{w}{R}$$

$$\epsilon_{xy}^0 = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

$$\kappa_x = \frac{\partial^2 w}{\partial x^2} - \frac{\partial \bar{\gamma}_{xz}}{\partial x}$$

(Continued)

$$\kappa_y = \frac{\partial^2 w}{\partial y^2} - \frac{\partial \bar{\gamma}_{yz}}{\partial y}$$

$$\kappa_{xy} = 2\,\frac{\partial^2 w}{\partial y^2} - \frac{\partial \bar{\gamma}_{xz}}{\partial y} - \frac{\partial \bar{\gamma}_{yz}}{\partial x}$$

Substitution of equations (11) and (12) into (8) and subsequent substitution of the result into equations (27) yields equations of the form

$$\begin{aligned} L_1 u + L_2 v &= L_9 w - L_4 \bar{\gamma}_{xz} - L_5 \bar{\gamma}_{yz} \\ L_2 u + L_6 v &= L_{10} w - L_5 \bar{\gamma}_{xz} - L_8 \bar{\gamma}_{yz} \end{aligned} \tag{32}$$

where the "L's" are linear operators defined as follows:

$$\left.\begin{aligned}
L_1 &= \left\{ A_{11} \frac{\partial^2}{\partial x^2} + 2A_{16} \frac{\partial^2}{\partial x \partial y} + A_{66} \frac{\partial^2}{\partial y^2} \right\} \\
L_2 &= \left\{ A_{16} \frac{\partial^2}{\partial x^2} + (A_{12} + A_{66}) \frac{\partial^2}{\partial x \partial y} + A_{26} \frac{\partial^2}{\partial y^2} \right\} \\
L_3 &= \left\{ D_{11} \frac{\partial^3}{\partial x^3} + (D_{12} + 2D_{66}) \frac{\partial^3}{\partial x \partial y^2} + 3D_{16} \frac{\partial^3}{\partial x^2 \partial y} + D_{26} \frac{\partial^3}{\partial y^3} \right\} \\
L_4 &= \left\{ D_{11} \frac{\partial^2}{\partial x^2} + 2D_{16} \frac{\partial^2}{\partial x \partial y} + D_{66} \frac{\partial^2}{\partial y^2} \right\} \\
L_5 &= \left\{ D_{16} \frac{\partial^2}{\partial x^2} + (D_{12} + D_{66}) \frac{\partial^2}{\partial x \partial y} + D_{26} \frac{\partial^2}{\partial y^2} \right\}
\end{aligned}\right\}$$

(Continued)

$$L_6 = \left\{ A_{66} \frac{\partial^2}{\partial x^2} + 2A_{26} \frac{\partial^2}{\partial x \partial y} + A_{22} \frac{\partial^2}{\partial y^2} \right\}$$

$$L_7 = \left\{ D_{16} \frac{\partial^3}{\partial x^3} + 3D_{26} \frac{\partial^3}{\partial x \partial y^2} + (2D_{66} + D_{12}) \frac{\partial^3}{\partial x^2 \partial y} + D_{22} \frac{\partial^3}{\partial y^3} \right\}$$

$$L_8 = \left\{ D_{66} \frac{\partial^2}{\partial x^2} + 2D_{26} \frac{\partial^2}{\partial x \partial y} + D_{22} \frac{\partial^2}{\partial y^2} \right\}$$

$$L_9 = L_3 + \frac{A_{12}}{R} \frac{\partial}{\partial x} + \frac{A_{26}}{R} \frac{\partial}{\partial y}$$

$$L_{10} = L_7 + \frac{A_{16}}{R} \frac{\partial}{\partial x} + \frac{A_{22}}{R} \frac{\partial}{\partial y} \tag{33}$$

Solution of the simultaneous equations (32) gives for the reference-surface displacements

$$\Delta u = (L_6 L_9 - L_2 L_{10}) w - (L_6 L_4 - L_2 L_5) \bar{\gamma}_{xz} - (L_6 L_5 - L_2 L_8) \bar{\gamma}_{yz}$$

$$\Delta v = (L_1 L_{10} - L_2 L_9) w - (L_1 L_5 - L_2 L_4) \bar{\gamma}_{xz} - (L_1 L_8 - L_2 L_5) \bar{\gamma}_{yz} \tag{34}$$

where

$$\Delta = L_1 L_6 - L_2^2$$

APPENDIX III

REFERENCE-SURFACE STRAIN COMPATIBILITY

Elimination of the reference surface-displacements between the strain-displacement relations given by equations (11) results in the following strain compatibility expression:

$$\frac{\partial^2 \epsilon_x^0}{\partial y^2} + \frac{\partial^2 \epsilon_y^0}{\partial x^2} - \frac{\partial^2 \epsilon_{xy}^0}{\partial x \partial y} = -\frac{1}{R}\frac{\partial^2 w}{\partial x^2} \tag{35}$$

Consideration of the equilibrium conditions for the x- and y-directions given by equation (18) allows the definition of the Airy function $\Phi(x,y)$ such that

$$N_x = \frac{\partial^2 \Phi}{\partial y^2}$$

$$N_y = \frac{\partial^2 \Phi}{\partial x^2} \tag{36}$$

$$N_{xy} = -\frac{\partial^2 \Phi}{\partial x \partial y}$$

With the constitutive relation given by equation (13) and the relations (36), the compatibility equation (35) becomes

$$a_{11}\Phi,_{yyyy} + 2a_{12}\Phi,_{xxyy} - 2a_{16}\Phi,_{xyyy} + a_{22}\Phi,_{xxxx} - 2a_{26}\Phi,_{xxxy}$$

$$+ a_{66}\Phi,_{xxyy} = (2d'_{66} - d'_{22} - d'_{11})\frac{\partial^4 w}{\partial x^2 \partial y^2} - d'_{12}\frac{\partial^4 w}{\partial y^4} - d'_{21}\frac{\partial^4 w}{\partial x^4}$$

$$(d'_{26} - 2d'_{16})\frac{\partial^4 w}{\partial x \partial y^3} + (d'_{61} - 2d'_{26})\frac{\partial^4 w}{\partial x^3 \partial y} + (d'_{11} - d'_{66})\frac{\partial^3 \bar{\gamma}_{xz}}{\partial x \partial y^2}$$

(Continued)

$$+ d'_{16} \frac{\partial^3 \bar{\gamma}_{xz}}{\partial y^3} + d'_{21} \frac{\partial^3 \bar{\gamma}_{xz}}{\partial x^3} + (d'_{26} - d'_{61}) \frac{\partial^3 \bar{\gamma}_{xz}}{\partial x^2 \partial y}$$

$$+ d'_{12} \frac{\partial^3 \bar{\gamma}_{yz}}{\partial y^3} + (d'_{16} - d'_{62}) \frac{\partial^3 \bar{\gamma}_{yz}}{\partial x \partial y^2} - (d'_{22} + d'_{66}) \frac{\partial^3 \bar{\gamma}_{yz}}{\partial x^2 \partial y}$$

$$+ d'_{26} \frac{\partial^3 \bar{\gamma}_{yz}}{\partial x^3} - \frac{1}{R} \frac{\partial^2 w}{\partial x^2} \tag{37}$$

APPENDIX IV

BUCKLING LOAD CALCULATION

DIRECT VARIATIONAL METHOD

Substitution of the assumed trigonometric functions given by equations (22), (23), and (24) into the functional (17) and use of the following integrations

$$\int_0^L \sin^2 \frac{m\pi x}{L}\, dx = \frac{L}{2}$$

$$\int_0^L \sin \frac{m\pi x}{L} \cos \frac{m\pi x}{L}\, dx = 0$$

$$\int_0^L \cos^2 \frac{m\pi x}{L}\, dx = \frac{L}{2}$$

$$\int_0^L \sin \frac{m\pi x}{L}\, dx = \left| 1 - (-1)^m \right| \frac{L}{m\pi}$$

$$\int_0^L \cos \frac{m\pi x}{L}\, dx = 0$$

as well as corresponding integrations with respect to y, yields

$$\begin{aligned}
U'' + V'' = & - N_3 u_2 \left(\frac{m\pi}{L}\right) \frac{Lb}{4} - N_1 v_1 \left(\frac{n\pi}{b}\right) \frac{Lb}{4} - N_1 \frac{W}{R} \frac{Lb}{4} \\
& + N_2 u_2 \left(\frac{n\pi}{b}\right) \frac{Lb}{4} + N_2 v_1 \left(\frac{m\pi}{L}\right) \frac{Lb}{4} + \frac{1}{2} \bar{G}_x h \Gamma_x^2 \frac{Lb}{4} + \frac{1}{2} \bar{G}_y h \Gamma_y^2 \frac{Lb}{4} \\
& + d_{11} N_3 W \left(\frac{m\pi}{L}\right)^2 \frac{Lb}{4} - d_{11} N_3 \Gamma_x \left(\frac{m\pi}{L}\right) \frac{Lb}{4} + d_{12} N_1 W \left(\frac{m\pi}{L}\right)^2 \frac{Lb}{4} \\
& - d_{12} N_1 \Gamma_x \left(\frac{m\pi}{4}\right) \frac{Lb}{4} + d_{21} N_3 W \left(\frac{n\pi}{b}\right)^2 \frac{Lb}{4} - d_{21} N_3 \Gamma_y \left(\frac{n\pi}{b}\right) \frac{Lb}{4} \\
& + d_{22} N_1 W \left(\frac{n\pi}{b}\right)^2 \frac{Lb}{4} - d_{22} N_1 \Gamma_y \left(\frac{n\pi}{b}\right) \frac{Lb}{4} - d_{66} N_2 2W \left(\frac{m\pi}{L}\right) \left(\frac{n\pi}{b}\right) \frac{Lb}{4}
\end{aligned}$$

(Continued)

$$+ d_{66}N_2\Gamma_x\left(\frac{n\pi}{b}\right)\frac{Lb}{4} + d_{66}N_2\Gamma_y\left(\frac{m\pi}{L}\right)\frac{Lb}{4}$$

$$- \frac{1}{2}a_{11}N_{x0}^2 Lb - \frac{1}{2}a_{11}N_3^2\frac{Lb}{4} - \frac{1}{2}a_{22}N_1^2\frac{Lb}{4} - \frac{1}{2}a_{66}N_2^2\frac{Lb}{4}$$

$$- a_{12}N_1N_3\frac{Lb}{4}$$

$$+ \frac{1}{2}d_{11}^*W^2\left(\frac{m\pi}{L}\right)^4\frac{Lb}{4} - d_{11}^*W\Gamma_x\left(\frac{m\pi}{L}\right)^3\frac{Lb}{4} + \frac{1}{2}d_{11}^*\Gamma_x^2\left(\frac{m\pi}{L}\right)^2\frac{Lb}{4}$$

$$+ \frac{1}{2}d_{22}^*W^2\left(\frac{n\pi}{b}\right)^4\frac{Lb}{4} - d_{22}^*W\Gamma_y\left(\frac{n\pi}{b}\right)^3\frac{Lb}{4} + \frac{1}{2}d_{22}^*\Gamma_y^2\left(\frac{n\pi}{b}\right)^2\frac{Lb}{4}$$

$$+ \frac{1}{2}d_{66}^*\Bigg[4W^2\left(\frac{m\pi}{L}\right)^2\left(\frac{n\pi}{b}\right)^2 + \Gamma_x^2\left(\frac{n\pi}{b}\right)^2 + \Gamma_y^2\left(\frac{m\pi}{L}\right)^2$$

$$- 4W\Gamma_x\frac{m\pi}{L}\left(\frac{n\pi}{b}\right)^2 - 4W\Gamma_y\left(\frac{m\pi}{L}\right)^2\left(\frac{n\pi}{b}\right) + 2\Gamma_x\Gamma_y\left(\frac{n\pi}{b}\right)\left(\frac{m\pi}{L}\right)\Bigg]$$

$$+ d_{12}^*W^2\left(\frac{m\pi}{L}\right)^2\left(\frac{n\pi}{b}\right)^2\frac{Lb}{4} - d_{12}^*W\Gamma_y\left(\frac{m\pi}{L}\right)^2\left(\frac{n\pi}{b}\right)\frac{Lb}{4} - d_{12}^*\Gamma_xW\left(\frac{n\pi}{b}\right)^2\left(\frac{m\pi}{L}\right)\frac{Lb}{4}$$

$$+ d_{12}^*\Gamma_x\Gamma_y\left(\frac{m\pi}{L}\right)\left(\frac{n\pi}{b}\right)\frac{Lb}{4} - \frac{1}{2}N_{x_o}W^2\left(\frac{m\pi}{L}\right)^2\frac{Lb}{4} \tag{38}$$

The variation of $U'' + V''$ with respect to the free parameters gives

$$\delta_W(U'' + V'') = \frac{Lb}{4}\Bigg[\Bigg\{d_{11}^*\left(\frac{m\pi}{L}\right)^4 + d_{22}^*\left(\frac{n\pi}{b}\right)^4 + 4d_{66}^*\left(\frac{m\pi}{L}\right)^2\left(\frac{n\pi}{b}\right)^2$$

$$+ 2d_{12}^*\left(\frac{m\pi}{L}\right)^2\left(\frac{n\pi}{b}\right)^2 - N_{x_o}\left(\frac{m\pi}{L}\right)^2\Bigg\}W$$

$$+ \left\{- d_{11}^*\left(\frac{m\pi}{L}\right)^3 - 2d_{66}^*\left(\frac{m\pi}{L}\right)\left(\frac{n\pi}{b}\right)^2 - d_{12}^*\left(\frac{n\pi}{b}\right)^2\left(\frac{m\pi}{L}\right)\right\}\Gamma_x$$

$$+ \left\{- d_{22}^*\left(\frac{n\pi}{b}\right)^3 - 2d_{66}^*\left(\frac{m\pi}{L}\right)^2\left(\frac{n\pi}{b}\right) - d_{12}^*\left(\frac{m\pi}{L}\right)^2\left(\frac{n\pi}{b}\right)\right\}\Gamma_y$$

$$+ \left\{- \frac{1}{R} + d_{12}\left(\frac{m\pi}{L}\right)^2 + d_{22}\left(\frac{n\pi}{b}\right)^2\right\}N_1 + \left\{- 2d_{66}\left(\frac{m\pi}{L}\right)\left(\frac{n\pi}{b}\right)\right\}N_2$$

$$+ \left\{d_{11}\left(\frac{m\pi}{L}\right)^2 + d_{21}\left(\frac{n\pi}{b}\right)^2\right\}N_3\Bigg] = 0 \tag{39a}$$

$$\delta_{\Gamma_x}(U'' + V'') = \frac{Lb}{4}\left[\left\{- d^*_{11}\left(\frac{m\pi}{L}\right)^3 - 2d^*_{66}\left(\frac{m\pi}{L}\right)\left(\frac{n\pi}{b}\right)^2 - d_{12}\left(\frac{n\pi}{b}\right)^2\left(\frac{m\pi}{L}\right)\right\} W\right.$$

$$+ \left\{\overline{G}_x h + d^*_{11}\left(\frac{m\pi}{L}\right)^2 + d^*_{66}\left(\frac{n\pi}{b}\right)^2\right\} \Gamma_x + \left\{d^*_{66}\left(\frac{n\pi}{b}\right)\left(\frac{m\pi}{L}\right) + d^*_{12}\left(\frac{m\pi}{L}\right)\left(\frac{n\pi}{b}\right)^2\right\}\Gamma_y$$

$$\left. + \left\{- d_{12}\left(\frac{m\pi}{L}\right)\right\} N_1 + \left\{d_{66}\left(\frac{n\pi}{b}\right)\right\} N_2 + \left\{- d_{11}\left(\frac{m\pi}{L}\right)\right\} N_3\right] = 0 \qquad (39b)$$

$$\delta_{\Gamma_y}(U'' + V'') = \frac{Lb}{4}\left[\left\{- d^*_{22}\left(\frac{n\pi}{b}\right)^3 - 2d^*_{66}\left(\frac{m\pi}{L}\right)^2\left(\frac{n\pi}{b}\right) - d^*_{12}\left(\frac{m\pi}{L}\right)^2\left(\frac{n\pi}{b}\right)\right\} W\right.$$

$$+ \left\{d^*_{66}\left(\frac{n\pi}{b}\right)\left(\frac{m\pi}{L}\right) + d^*_{12}\left(\frac{m\pi}{L}\right)\left(\frac{n\pi}{b}\right)\right\}\Gamma_x + \left\{\overline{G}_y h + d^*_{22}\left(\frac{n\pi}{b}\right)^2 + d^*_{66}\left(\frac{m\pi}{L}\right)^2\right\}\Gamma_y$$

$$\left. + \left\{- d_{22}\left(\frac{n\pi}{b}\right)\right\} N_1 + \left\{d_{66}\left(\frac{m\pi}{L}\right)\right\} N_2 + \left\{- d_{21}\left(\frac{n\pi}{b}\right)\right\} N_2\right] = 0 \qquad (39c)$$

$$\delta_{u_1}(U'' + V'') = 0 \qquad (39d)$$

$$\delta_{u_2}(U'' + V'') = \frac{Lb}{4}\left[- \left(\frac{m\pi}{L}\right) N_3 + \left(\frac{n\pi}{b}\right) N_2\right] = 0 \qquad (39e)$$

$$\delta_{v_1}(U'' + V'') = \frac{Lb}{4}\left[- \left(\frac{n\pi}{b}\right) N_1 + \left(\frac{m\pi}{L}\right) N_2\right] = 0 \qquad (39f)$$

$$\delta_{v_2}(U'' + V'') = 0 \qquad (39g)$$

$$\delta_{N_3}(U'' + V'') = \frac{Lb}{4}\left[\left\{d_{11}\left(\frac{m\pi}{L}\right)^2 + d_{21}\left(\frac{n\pi}{b}\right)^2\right\} W - d_{11}\left(\frac{m\pi}{L}\right)\Gamma_x\right.$$

$$\left. - d_{21}\left(\frac{n\pi}{b}\right)\Gamma_y - \left(\frac{m\pi}{L}\right)u_2 - a_{12}N_1 - a_{11}N_3\right] = 0 \quad (39h)$$

$$\delta_{N_1}(U'' + V'') = \frac{Lb}{4}\left[\left\{-\frac{1}{R} + d_{12}\left(\frac{m\pi}{L}\right)^2 + d_{22}\left(\frac{n\pi}{b}\right)^2\right\} W - d_{12}\left(\frac{m\pi}{L}\right)\Gamma_x\right.$$

$$\left. - d_{22}\left(\frac{n\pi}{b}\right)\Gamma_y - \left(\frac{n\pi}{b}\right)v_1 - a_{22}N_1 - a_{12}N_3\right] = 0 \quad (39i)$$

$$\delta_{N_2}(U'' + V'') = \frac{Lb}{4}\left[-2d_{66}\left(\frac{m\pi}{L}\right)\left(\frac{n\pi}{b}\right)W + d_{66}\left(\frac{n\pi}{b}\right)\Gamma_x + d_{66}\left(\frac{m\pi}{L}\right)\Gamma_y\right.$$

$$\left. + \left(\frac{n\pi}{b}\right)u_2 + \left(\frac{m\pi}{L}\right)v_1 - a_{66}N_2\right] = 0 \quad (39j)$$

Through the use of equations (39e), (39f), (39h), and (39i) to eliminate N_3, N_1, N_2, and v_1, the system of equations (39) is reduced quite readily to a set of linear, homogeneous equations in terms of W, Γ_x, Γ_y, and N_2. A non-trivial solution of the set of four equations exists only if the determinant of the coefficients vanishes; that is,

$$\begin{bmatrix} (\alpha + \beta + 2\gamma) + \frac{\bar{N}_{x_o}}{N^2} & -\lambda_x(\alpha + \gamma) & -\lambda_y(\beta + \gamma) & -\left\{\frac{1}{N^2} + (\nu + \xi)\right\} \\ -\lambda_x(\alpha + \gamma) & \frac{1}{N^2} + \lambda_x^2\alpha & \lambda_x\lambda_y\gamma & \lambda_x\nu \\ -\lambda_y(\beta + \gamma) & \lambda_x\lambda_y\gamma & \frac{1}{N^2\mu^2} + \lambda_y^2\beta & \lambda_y\xi \\ -\left\{\frac{1}{N^2} + (\nu + \xi)\right\} & \lambda_x\nu & \lambda_y\xi & \kappa \end{bmatrix} = 0 \quad (40)$$

where the various nondimensional parameters are defined as follows:

$$
\left.
\begin{aligned}
\alpha &= \frac{\mu^2 d_{11}^* + d_{66}^*}{d_{11}^*} \\
\beta &= \frac{(1/\mu^2)\, d_{22}^* + d_{66}^*}{d_{11}^*} \\
\gamma &= \frac{d_{12}^* + d_{66}^*}{d_{11}^*} \\
\nu &= \frac{-\, d_{11} - \mu^2 d_{12} + d_{66}}{R} \\
\xi &= \frac{-\, d_{22} - (1/\mu^2)\, d_{21} + d_{66}}{R} \\
\kappa &= -\,\frac{d_{11}^* \left[2a_{12} + (1/\mu^2)\, a_{11} + \mu^2 a_{22} + a_{66}\right]}{R^2} \\
\lambda_x &= \left(\frac{d_{11}^*}{R^2 h\, \bar{G}_x}\right)^{1/2} \\
\lambda_y &= \left(\frac{d_{11}^*}{R^2 h\, \bar{G}_y}\right)^{1/2}
\end{aligned}
\right\} \qquad (41)
$$

(Continued)

$$\bar{N}_{x_o} = \frac{N_{x_o} R^2}{d_{11}^*}$$

$$\mu = \left(\frac{m\pi}{L}\right) \Big/ \left(\frac{n\pi}{b}\right)$$

APPENDIX V

VIBRATION PROBLEM

The governing equations for the free lateral vibration of the circular cylindrical shell can be derived using the following form of Hamilton's variational principle:

$$\delta \int_{t_1}^{t_2} (U'' - T'')\, dt = 0 \tag{42}$$

In equation (42), U'' represents the Reissner form of the strain energy and T'' represents the kinetic energy. In this case, both U'' and T'' are functions of time. With the assumption of harmonic motion, the time dependence of equation (42) is eliminated and the variational principle is stated as

$$\delta(U'' - T'') = 0 \tag{43}$$

For the fiber-reinforced shell, U'' is given in equation (17). When the radial (lateral) motion of the shell is assumed to predominate, T'' is given by

$$T'' = -\frac{\rho\omega^2}{2} \int_0^b \int_0^L w^2\, dxdy \tag{44}$$

where ω is the natural frequency and ρ is the mass density of the shell per unit reference surface area. The functional for the variational procedure is expressed as

$$U'' - T'' = \iint_A \left[N_x \frac{\partial u}{\partial x} + N_y \left(\frac{\partial v}{\partial y} - \frac{w}{R} \right) + N_{xy} \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) + \frac{1}{2} \bar{G}_x h \bar{\gamma}_{xz}^2 \right.$$

$$\left. + \frac{1}{2} \bar{G}_y h \bar{\gamma}_{yz}^2 \right.$$

(Continued)

$$
\begin{aligned}
&- (d_{11}N_x + d_{12}N_y + d_{16}N_{xy})\left(\frac{\partial^2 w}{\partial x^2} - \frac{\partial \bar{\gamma}_{xz}}{\partial x}\right) \\
&- (d_{21}N_x + d_{22}N_y + d_{26}N_{xy})\left(\frac{\partial^2 w}{\partial y^2} - \frac{\partial \bar{\gamma}_{yz}}{\partial y}\right) \\
&- (d_{61}N_x + d_{62}N_y + d_{66}N_{xy})\left(2\frac{\partial^2 w}{\partial x \partial y} - \frac{\partial \bar{\gamma}_{xz}}{\partial y} - \frac{\partial \bar{\gamma}_{yz}}{\partial x}\right) \\
&- \frac{1}{2}(a_{11}N_x^2 + a_{22}N_y^2 + a_{66}N_{xy}^2 + 2a_{12}N_xN_y + 2a_{16}N_xN_{xy} + 2a_{26}N_yN_{xy}) \\
&+ \frac{1}{2}\Bigg\{d_{11}^*\left(\frac{\partial^2 w}{\partial x^2} - \frac{\partial \bar{\gamma}_{xz}}{\partial x}\right)^2 + d_{22}^*\left(\frac{\partial^2 w}{\partial y^2} - \frac{\partial \bar{\gamma}_{yz}}{\partial y}\right)^2 + d_{66}^*\left(2\frac{\partial^2 w}{\partial x \partial y} - \frac{\partial \bar{\gamma}_{xz}}{\partial y} - \frac{\partial \bar{\gamma}_{yz}}{\partial x}\right)^2 \\
&+ 2d_{12}^*\left(\frac{\partial^2 w}{\partial x^2} - \frac{\partial \bar{\gamma}_{xz}}{\partial x}\right)\left(\frac{\partial^2 w}{\partial y^2} - \frac{\partial \bar{\gamma}_{yz}}{\partial y}\right) \\
&+ 2d_{16}^*\left(\frac{\partial^2 w}{\partial x^2} - \frac{\partial \bar{\gamma}_{xz}}{\partial x}\right)\left(2\frac{\partial^2 w}{\partial x \partial y} - \frac{\partial \bar{\gamma}_{xz}}{\partial y} - \frac{\partial \bar{\gamma}_{yz}}{\partial x}\right) \\
&+ 2d_{26}\left(\frac{\partial^2 w}{\partial y^2} - \frac{\partial \bar{\gamma}_{yz}}{\partial y}\right)\left(2\frac{\partial^2 w}{\partial x \partial y} - \frac{\partial \bar{\gamma}_{xz}}{\partial y} - \frac{\partial \bar{\gamma}_{yz}}{\partial x}\right)\Bigg]\Bigg\}dxdy \\
&+ \frac{\rho\omega^2}{2}\iint_A w^2 dxdy
\end{aligned}
\tag{45}
$$

Variation of the functional given by equation (45) with respect to N_x, N_y, N_{xy}, u, v, w, $\bar{\gamma}_{xz}$, $\bar{\gamma}_{yz}$ yields the Euler equations and associated boundary conditions for the vibration problem. If N_{x_o} is set equal to zero, the boundary conditions and the Euler equations, except for the lateral equilibrium equation, are identical to those established in Appendix I for the buckling problem. The lateral

equilibrium equation for the vibration problem becomes

$$\frac{\partial^2 M_x}{\partial x^2} + 2\frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} = -\frac{N_y}{R} + \rho\omega^2 w \tag{46}$$

With the use of the assumed generalized displacements and forces given by equations (22), (23), and (24) with N_{x_o} set equal to zero, the direct variational procedure yields a frequency equation in determinant form as

$$\begin{vmatrix} (\alpha+\beta+2\gamma)+\frac{\bar{\omega}}{N^4\mu^2} & -\lambda_x(\alpha+\gamma) & -\lambda_y(\beta+\gamma) & -\left\{\frac{1}{N^2}+(\nu+\epsilon)\right\} \\ -\lambda_x(\alpha+\gamma) & \frac{1}{N^2}+\lambda_x^2\alpha & \lambda_x\lambda_y\gamma & \lambda_x\nu \\ -\lambda_y(\beta+\gamma) & \lambda_x\lambda_y\gamma & \frac{1}{N^2\mu^2}+\lambda_y^2\beta & \lambda_y\epsilon \\ -\left\{\frac{1}{N^2}+(\nu+\epsilon)\right\} & \lambda_x\nu & \lambda_y\epsilon & \kappa \end{vmatrix} = 0 \tag{47}$$

where

$$\bar{\omega} = \frac{\rho\omega^2 R^4}{d_{11}^*}$$

APPENDIX VI

DEFINITIONS OF MATRICES

$$[A] = \begin{bmatrix} A_{11} & A_{12} & A_{16} \\ A_{12} & A_{22} & A_{26} \\ A_{16} & A_{26} & A_{66} \end{bmatrix} \qquad (48)$$

where

$$A_{ij} = \sum_{k=1} C_{ij}(h_{k+1} - h_k)$$

$$[D] = \begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{16} & D_{26} & D_{66} \end{bmatrix} \qquad (49)$$

where

$$D_{ij} = \frac{1}{2} \sum_{k=1} C_{ij}(h_{k+1}^2 - h_k^2)$$

$$D^* = \begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{16} & D_{26} & D_{66} \end{bmatrix} \qquad (50)$$

where

$$D_{ij}^* = \frac{1}{3} \sum_{k=1} C_{ij}(h_{k+1}^3 - h_k^3)$$

$$[a] = [A]^{-1} \tag{51}$$

$$[d'] = [a][D] \tag{52}$$

$$[d] = [D][a] = [d']^{T} \tag{53}$$

$$[d^{*}] = [D^{*}]-[D][a][D] = [D^{*}]-[D][d'] \tag{54}$$

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13. ABSTRACT

The effects of transverse shear deformations upon the classical buckling load of fiber-reinforced cylindrical shells under uniform axial compression have been analyzed. A stability determinant which upon evaluation yields an expression for the buckling load has been developed using a modified form of the Reissner variational principle. The buckling loads predicted by the stability determinant, with transverse shear effects neglected, have been calculated and shown to agree with previously published results for boron-epoxy and glass-epoxy cylinders. Inclusion of the transverse shear effects in the two cases investigated shows little reduction in the classical buckling loads. For general application, charts are presented to give stability criteria for fiber-reinforced composites as a function of the geometric and mechanical properties. Effective transverse shear rigidities, established by experiment, must be developed in order to realistically estimate the reduction in buckling load from the charts presented. For the determination of natural frequencies of fiber-reinforced shells, with transverse shear effects included, a frequency equation in determinant form is presented.

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